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13. ABSTRACT (Maximum 200 words)
   Problems in harmonic analysis and synthesis are intertwined with their applications in signal and image processing. Two direct applications of these theories are in the development of multichannel deconvolution and parameter estimation. The first is the recovery of information from linear, translation invariant systems. The work in this area has progressed in a variety of areas. The general theory, and its relationship to wavelet and Gabor analysis was developed. A very general method for the creation of strongly coprime deconvolvers was constructed, and the interaction of deconvolution and sampling was developed. Properly sampled signals can be deconvolved and reconstructed as analog signals simultaneously. Sampling theory also provided a new method for constructing deconvolvers. Work was done on "deconvolving Wiener filters." Very simply, the main idea of this work is to optimally deconvolve a signal from a noisy environment. The theory was extented by increasing the types of systems that can be modeled, including linear combinations of n-fold convolutions of characteristic functions with equally spaced knots (cardinal splines), and truncated sinc and truncated Gaussian functions. Finally, the theory was simulated, providing the first step in linking the theory to its many potential applications.
   The second area is in the develpment of computationally straightforward and very general algorithms for several classes of problems in estimation. Particular items investigated included development of computationally straightforward techniques for simple spectral analysis of a very broad class of periodic processes, and the extention of these techniques to the analysis of data generated by multiple periodic generators, including the deinterleaving of these generators. This work is may be applied to communications systems, radar and sonar, biomedical systems, etc.

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Final Technical Report on
Multichannel Deconvolution with Applications to Signal and Image Processing

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1.) OBJECTIVES

Problems in harmonic analysis and synthesis are intertwined with their applications in signal and image processing. Some recent advances in the theory of this analysis have used number theory to extend existing theories (e.g., sampling theory, fast computations) and develop new approaches to problems (e.g., interpolation). Two direct applications of these theories are in the development of multichannel deconvolution and parameter estimation.

The first area which has benefited from these new techniques is the recovery of information from linear, translation invariant systems. This approach has been labeled "multichannel deconvolution." Assuming system impulse response function μ, these systems are modeled by the convolution equation $s = f * \mu$. The output s may be an inadequate approximation of f, which motivates solving the convolution equation for f, i.e. deconvolving f from μ. If the function μ is time–limited (compactly supported) and non–singular, this deconvolution problem is ill–posed. A theory of solving such equations has been developed. It circumvents ill–posedness by using a multichannel system. If we overdetermine the signal f by using a system of convolution equations,

$$s_i = f * \mu_i \ , \ i = 1, \ldots, n,$$

the problem of solving for f is well–posed if the set of convolvers $\{\mu_i\}$ satisfies the *strongly coprime* condition. In this case, there exist compactly supported distributions (deconvolvers) ν_i , $i = 1, \ldots, n$ such that

$$1 = \widehat{\mu}_1 \cdot \widehat{\nu}_1 + \ldots + \widehat{\mu}_n \cdot \widehat{\nu}_n \,,$$

the Bezout equation. Transforming, we get

$$\delta = \mu_1 * \nu_1 + \ldots + \mu_n * \nu_n \,,$$

which in turn gives

$$f = s_1 * \nu_1 + \ldots + s_n * \nu_n \,.$$

We look to continue the development of the theory. Particular items include:

i.) Extend the theory to more general convolvers, including linear combinations of characteristic functions, linear combinations of n–fold convolutions of characteristic functions with equally spaced knots (cardinal splines), and truncated sinc and truncated Gaussian functions.

ii.) Fit our general models of system kernels with models of currently deployed active and passive remote sensors as convolution equations.

iii.) Coordinate the theory with sampling theory.

iv.) Use the theory to create filters.

v.) Coordinate the theory with wavelet and Gabor analysis.

vi.) Apply of the theory to image and speech processing, where the deconvolvers would act as preliminary enhancement operators.

The second area that has benefited from the blend of number theory and harmonic analysis is that of parameter estimation. In particular, developments in the theory on the zeta function and algorithms on Euclidean domains may lead to extremely efficient and very general algorithms for several classes of problems in estimation. Particular items to be investigated include:

vii.) Development of computationally straightforward techniques for spectral analysis of a very broad class of periodic processes. Include procedures so that estimates achieve the Cramer–Rao bound.

viii.) Extend these techniques to the complete analysis of periodic point processes, including the recovery of the fundamental period(s), phase information, the multiples of the periods, and the deinterleaving of the data.

The objectives of the grant have changed in that the mathematical techniques used in the multichannel deconvolution theory were directly applicable to the problems in parameter estimation. This lead to solutions of a whole class of these problems including the estimation of the period, phase, and period multiples. This work is directly applicable to communications systems, radar and sonar, biomedical systems, etc. (see **4.**)).

2.) STATUS OF EFFORT

Various items in the objectives list have been addressed. The technical content of this work is described in items **3.)** and **7.)** below.

The work in multichannel deconvolution has progressed in a variety of areas. The general theory, and its relationship to Shannon sampling and wavelet and Gabor analysis was developed in the *SIAM Review* paper. This paper was accepted "as is," and has been widely read, as is evidenced by the nearly 200 requests for offprints received by the authors. The paper submitted to *Inverse and Ill–Posed Problems* contains a very general method for the creation of strongly coprime deconvolvers, and fully develops the interaction of deconvolution and sampling. Properly sampled signals can be deconvolved and reconstructed as analog signals simultaneously. In addition, sampling theory is used to provide a new method for constructing deconvolvers. These papers address **i.)**, **iii.)**, **v.)**.

Work on three additional areas is nearly completed. The first is work on developing "deconvolving Wiener filters." Theoretical developments in this area go a long way towards the development of actual multichannel systems. Very simply, the main idea of this work is to optimally deconvolve (in the sense of L^2) a signal from a noisy environment. The research for this paper is complete. The paper is being written up this fall. The second is simulation of the theory. This work was done in coordination with a graduate student (Matthew Fivash). Excellent simulations were developed. These provided the first step in linking the theory to its many potential applications. A journal version of these results is in preparation. The third area is the extension of the theory by increasing

the types of systems that can be modeled by the multichannel theory. A considerable amount of this research work is completed. These papers address **i.)**, **iii.)**, **iv.)**, **v.)**, and provide a basis for addressing **ii.)**, **vi.)**.

In developing the theory, we discovered that techniques used in the multichannel deconvolution theory were directly applicable to a broad class of problems in parameter estimation. This lead to solutions of a whole class of these problems, including the estimation of the period, phase, and phase multiples. This work is directly applicable to communications systems, radar and sonar, biomedical systems, etc. The last two items in the objectives list above have been essentially completed. Complete results for **vii.)** will appear in *IEEE Transactions on Signal Processing*, and were presented and very well received at *ICASSP '95*. This work provides computationally straightforward procedures for period estimation for very general types of periodic point processes. These techniques can be applied directly to pulse repetition interval (PRI) analysis from time–of–arrival measurements. These particular applications were presented in an invited paper at *Asilomar '95*. Results for **viii.)** have been submitted to *IEEE Transactions on Signal Processing*, and have been accepted for presentation at *ICASSP '96*. These last three papers include multiple step procedures for estimation that achieves the Cramer–Rao bound.

3.) ACCOMPLISHMENTS/NEW FINDINGS

i.) Systems of Convolution Equations, Deconvolution, Shannon Sampling, and the Wavelet and Gabor Transforms

SIAM Review (December 1994)

Linear, translation invariant systems (e.g., sensors, linear filters) are modeled by the convolution equation $s = f * \mu$, where f is the input signal, μ is the system impulse response function (or, more generally, impulse response distribution), and s is the output signal. In many applications, the output s is an inadequate approximation of f, which motivates solving the convolution equation for f, i.e. deconvolving f from μ. If the function μ is time–limited (compactly supported) and non–singular, it is shown that this deconvolution problem is ill–posed.

A theory of solving such equations has been developed by Berenstein et al. It circumvents ill–posedness by using a multichannel system. If the signal f is overdetermined by using a system of convolution equations. $s_i = f * \mu_i$, $i = 1, \ldots, n$, the problem of solving for f is well–posed if the set of convolvers $\{\mu_i\}$ satisfies the condition of being what is called strongly coprime. In this case, there exist compactly supported distributions (deconvolvers) ν_i , $i = 1, \ldots, n$ such that

$$\delta = \mu_1 * \nu_1 + \ldots + \mu_n * \nu_n ,$$

which in turn gives

$$f = s_1 * \nu_1 + \ldots + s_n * \nu_n .$$

The strongly coprime condition is described and examples of sets of strongly coprime sensors and their deconvolutions are given.

It is then shown how the theory can work in conjunction with the Shannon Sampling Theorem. In particular, if it is assumed that f is band–limited, then the analog signal f may be reconstructed from the sampled outputs of the sensors $\{f * \mu_i\}$ if the sampling rate is greater than or equal to Nyquist. To do this, new interpolating functions were created by combining sinc functions with the deconvolvers.

Since the purpose of deconvolution is greater signal resolution, it is shown how data compression and analysis using the wavelet and Gabor transforms are implemented in coordination with the deconvolution methods described above. Under the proposed procedure, wavelet or Gabor coefficients of the signal f may be recovered from $\{f * \mu_i\}$ in a single processing step.

This work provides a theoretical basis for multichannel systems in signal and image processing. It is indirectly applicable to AF technologies in that it provides new theoretical basis for how to develop active and passive remote sensing systems which gather all possible information, new techniques for making filters, etc. The multichannel approach turns ill–posed problems into well–posed ones, and provides a framework for solution by linear methods.

ii.) Bezout Equations, Multichannel Deconvolution and Sampling
submitted to *Inverse and Ill–Posed Problems*

The paper develops deconvolution procedures. It circumvents the ill–posedness inherent in deconvolution by using a multichannel system. The theory of deconvolution presented in this paper is contained in a larger group of results in the theory of residues of analytic functions and their generalities, for example, intersection varieties. These results have appeared in a series of papers by Berenstein, Gay, Taylor, Yger et al., and can be interpreted as results in division problems, interpolation of analytic functions, analytic continuation, digital to analog conversion, and complexity theory. For deconvolution and other applications to signal and image processing, the theory focuses on solving the general analytic Bezout equation, i.e., for given holomorphic f_i and ψ satisfying certain growth conditions, solving for holomorphic g_i satisfying the same growth conditions such that

$$f_1 \cdot g_1 + \ldots + f_n \cdot g_n = \psi.$$

In many situations, $\psi = \psi_\lambda$, with $\psi_\lambda \to 1$ as $\lambda \to \infty$ (ψ_λ is the transform of an approximate identity).

The strongly coprime condition is developed and examples of sets of strongly coprime system response functions and their deconvolutions for functions in one and several variables are given. New methods of creating strongly coprime systems by using modulation are developed, and the deconvolvers associated with these systems are constructed. In the language of applications, the set of convolvers $\{\mu_i\}$ models a linear translation invariant multichannel system consisting of an array of sensors or filters. The system is created so that *no* information contained in the input signal f is lost. When the modulation results are applicable, it may be possible to create the system without hardware modifications. The signal f is gathered by this system as $\{s_i = f * \mu_i\}$. The signals s_i are then filtered by the ν_i (which have been created digitally, optically, etc., in coordination with the creation of the system and possibly tailored to be optimized under some constraint) and added, resulting in the reconstruction of f. The various classes of impulse response functions of modeled by the theory are discussed.

It is shown how the theory can work in coordination with the Shannon Sampling Theorem. In particular, if it is assumed that f is band–limited, then the analog signal f may be reconstructed from the sampled outputs of the sensors $\{f * \mu_i\}$ if the sampling rate is greater than or equal to Nyquist. To do this, new interpolating functions are created by combining sinc functions with the deconvolvers. This is solved in the case of arbitrary coprime system response functions in one variable. Real–variable methods using sampling which solve Bezout are developed in specific cases.

This work provides a theoretical basis for multichannel systems in signal and image processing. It is also indirectly applicable to AF technologies in that it provides new theoretical basis for how

to develop active and passive remote sensing systems which gather all possible information, new techniques for making filters, etc. Again, the multichannel approach turns ill–posed problems into well–posed ones, and provides a framework for solution by linear methods.

iii.) Modifications of the Euclidean Algorithm for Isolating Periodicities From a Sparse Set of Noisy Measurements

to appear in *IEEE Transactions on Signal Processing, ICASSP '95* (May 1995)

Modifications of the Euclidean algorithm are presented for determining the period from a sparse set of noisy measurements. The elements of the set are the noisy occurrence times of a periodic event with (perhaps very many) missing measurements. This problem arises in radar pulse repetition interval (PRI) analysis, in bit synchronization in communications, and other scenarios. The proposed algorithms are computationally straightforward and converge quickly. A robust version is developed that is stable despite the presence of arbitrary outliers. The Euclidean algorithm approach is justified by a theorem which shows that, for a set of randomly chosen positive integers, the probability that they do not all share a common prime factor approaches one quickly as the cardinality of the set increases. The result is expressed in terms of the Riemann Zeta Function.

In the noise–free case our algorithm is equivalent to the Euclidean algorithm and converges with very high probability given only $n = 10$ data samples, independent of the number of missing measurements. Simulation examples demonstrate successful estimation of τ for $n = 10$ with 99.99% of the possible measurements missing. In fact, with only 10 data samples, it is possible to have the percentage of missing measurements arbitrarily close to 100%. There is, of course, a cost, in that the number of iterations the algorithm needs to converge increases with the percentage of missing measurements.

In the presence of noise and false data (or outliers), there is a tradeoff between the number of data samples, the amount of noise, and the percentage of outliers. The algorithm will perform well given low noise for $n = 10$, but will degrade as noise is increased. However, given more data, it is possible to reduce noise effects and speed up convergence by binning the data, and averaging across bins. Binning can be effectively implemented by using an adaptive threshhold with a gradient operator, allowing convergence in a single iteration in many cases. Simulation results show, for example, good estimation of the period from one hundred data samples with fifty percent of the measurements missing and twenty five percent of the data samples being arbitrary outliers.

This work is directly applicable to AF technologies in that it provides a new and surprisingly straightforward method for parameter estimation from radar and sonar data. It provides a computationally efficient procedure for extracting the period from a very general model of a periodic point process. It is also applicable to biomedical data and general communications data.

iv.) Pulse Interval Analysis From Sparse Data via a Modified Euclidean Algorithm

Asilomar '95 (October 1995)

This paper discusses effective parameter estimation in pulse interval analysis, with direct application to radar pulse repetition interval (PRI) analysis. The data is modeled as a sparse set of time–of–arrival measurements with additive jitter noise, and arbitrary outliers. We develop a three step procedure to analyze the data. The first is to apply a modified Euclidean algorithm for determining the PRI. This algorithm is a computationally simple, robust method for estimating the greatest common divisor of a noisy contaminated data set. The resulting estimate, while not maximum likelihood, is used as initialization in a three–step algorithm that achieves the Cramer–Rao

bound for moderate noise levels, as shown by comparing Monte Carlo results with the Cramer–Rao bounds.

This work is also directly applicable to AF technologies in that it provides a robust method for parameter estimation from radar and sonar data. It is also applicable to biomedical data and general communications data.

v.) On Periodic Pulse Interval Analysis with Outliers and Missing Observations
submitted to *IEEE Transactions on Signal Processing*

Analysis of pulse trains is a long standing problem, with applications including radar, communications, neurology, and astronomy. For example, in radar it may be desired to accurately estimate the pulse repetition interval (PRI). A similar problem is frequency estimation in additive white Gaussian noise at moderate to high signal to noise ratio (SNR) using phase data. At its most fundamental, pulse train analysis is based on time of arrival information only, as might be obtained from a matched filter or other detector. We assume time is highly resolved and ignore any time quantization error. In this work we are primarily concerned with a single periodic pulse train with (perhaps very many) missing observations that may be contaminated with outliers. Our data model for this case, in terms of the arrival times t_j, is given by

$$t_j = \phi + k_j T + v_j, \quad j = 1, \cdots, N,$$

where T is the unknown period, $\phi \sim \mathcal{U}[0, T)$ is a uniformly distributed random phase, the k_j's are positive non–repeating integers, and v_j is zero–mean additive white Gaussian noise. This model allows for missing observations through the distribution of the k_j's, e.g., $k_j = 1, 3, 5, \cdots$ implies observations at $k_j = 2, 4, \cdots$ are missing. Outliers are included in the data as phase–shifted multiples of another period. Alternative data models, commonly used in neurology, assume "integrate and fire" pulse generation. Cumulative error terms may also be included in some cases.

Given samples as in the data set above, the problem is to recover the period T and possibly the phase ϕ. The minimum variance unbiased estimates for this linear regression problem take a least–squares form. However, this requires knowledge of the k_j's. We therefore propose a multi-step procedure that proceeds by (i) estimating T directly, (ii) estimating the k_j's, and (iii) refining the estimate of T using the estimated k_j's in the least–squares solution. This estimate is shown to perform well, achieving the Cramer–Rao bound in many cases, despite many missing observations and contaminated data.

The direct estimate of T (step (i) above) is obtained using a modified Euclidean algorithm. This approach is motivated by the fact that, in the noise free case, the value of T is the greatest common divisor (gcd) of a set of samples $\{t_j - \phi\}_{j=1}^{N}$. The modified Euclidean algorithm is a computationally simple, robust method for estimating the gcd from a noisy contaminated data set.

The paper includes the following. We first give the maximum likelihood solution and Cramer–Rao bounds for estimating T and ϕ. Our analysis has led us to work with the data set $\{t_{j+1} - t_j\}_{j=1}^{N-1}$, so as to avoid estimating ϕ (which can be unreliable). For completeness, related work based on a point process (zero–one time series) viewpoint is briefly reviewed. Next, we describe our modified Euclidean algorithms for estimating T, including a robust version for use with contaminated data. Then, the three–step refined estimation procedure mentioned above is developed. Its performance is compared to Cramer–Rao bounds via Monte Carlo simulation, revealing the very good performance of the algorithm with many missing observations and contaminated data. We conclude with a discussion of the "common oscillator" problem and its solution using our approach.

This work is directly applicable to AF technologies in that it provides a robust method for parameter estimation from radar and sonar data. We give procedures for estimating the fundamental period, the phase, and the multiples of the fundamental period (the $\{k'_j s\}$) from the data. It is also applicable to biomedical data and general communications data.

vi.) Frequency Estimation Via Sparse Zero Crossings
to appear in *ICASSP '96* (May 1996)

Estimation of the frequency of a single complex sinusoid in Gaussian noise is a fundamental problem in signal processing. In this paper we address the problem, using only very sparse noisy zero–crossings with the presence of outliers. This paper develops procedures for simple spectral analysis that combine filtering and number–theoretic based methods. The basic procedure is to first filter noisy time–series data, and extract zero–crossing data from the series. This data is then analyzed by using modifications of the Euclidean algorithm to isolate the fundamental period of the data. This analysis takes advantage of the fact that these modified algorithms are computationally straightforward, converge quickly, and are robust in that they are stable despite the presence of noise and arbitrary outliers. This gives a procedure for performing a basic spectrum analysis on extremely noisy time–series data.

This work is directly applicable to AF technologies in that it provides a new procedure for the spectral analysis of sparse noisy time–series data. Thus, it is applicable to radar and sonar data. It is also applicable to biomedical data and general communications data.

vii.) Optimal Multichannel Deconvolution in a Noisy Environment: Deconvolving Wiener Filters
to be submitted to *Fourier Analysis and Applications*

This paper gives the development of optimal deconvolvers (in the sense of L^2) for a noisy signal environment. The noise assumed will be generated by a wide sense stationary process and added to both the signal and to each of the channels. The output of each channel is then convolved with a "deconvolving Wiener filter," and then added. It is shown that the estimated signal minimizes the mean square error with the original signal.

The deconvolving filters were developed by using the orthogonality principle and the non–uniqueness of the deconvolvers. The filters are the solutions to the inverse Fourier–Laplace transform of the analytic Bezout equation, given the constraint that they minimized the noise. The formulae for the filters are expressed in terms of the fundamental solutions to the Bezout equation , i. e., the solutions from a noise–free environment, and the power spectral density of the noise. Filters have been developed for deconvolvers in the Schwarz class and for approximate deconvolvers.

This work extends the theory of multichannel systems for signal and image processing. We show that these systems can be tuned in a reasonable noise environment to produce an optimal estimate of the original signal. It is indirectly applicable to AF technologies in that it provides new theoretical basis for the development of active and passive remote sensing systems which gather all possible information in a noisy environment, new techniques for making filters, etc.

viii.) Current Projects

Multichannel Deconvolution, Splines and Sampling

This paper broadens the class of linear translation invariant systems to which the multichannel theory is applicable. In particular, it extends the theory to more general convolvers, including

linear combinations of n-fold convolutions of characteristic functions with equally spaced knots (cardinal splines), and truncated sinc and truncated Gaussian functions. The paper also includes results which allow for development of deconvolvers by sampling theory.

Radios, Radar, and the Zeta Function of Riemann: With Apologies to Professor Hardy
to be submitted to *Mathematics Magazine*, 46 pp.

This is an expository article which develops the mathematical theory used in the development of the modified Euclidean and deinterleaving algorithms. A new proof is given for the theorem which shows that, for a set of randomly chosen positive integers, the probability that they do not all share a common prime factor approaches one quickly as the cardinality of the set increases. The result is expressed in terms of the Riemann Zeta Function. Additionally, by using an integral representation of the Zeta function, detailed convergence rates are derived. The paper also includes a discussion of Hardy's famous *Apology* in light of recent applications of number theory.

Multichannel Deconvolution: Theory and Procedure

The first portion of this paper gives an overview of the multichannel deconvolution theory. The paper then goes on to the details of simulations of this theory, including explicit estimates for truncation error, analysis of numerical noise, and techniques used to optimize computations.

New Methods for the Inversion of Radon Transforms

It is of interest to apply some of the techniques developed in the study of deconvolution to the Radon transform. A major problem in the theory and application of the Radon transform is the "the uniqueness problem" in the study of a transformed image. In theory, a compactly supported continuous function is uniquely determined. It is not, however, if the transform is sampled (as must be the case in any actual application). It may be possible to develop an irregular sampling (as was done for deconvolution) to uniquely determine the function.

ix.) Work on Fractal Geometry
IEEE Computer Graphics and Applications (May 1994), *Computer Graphics 1995* (February 1995), *The Pattern Book* (October 1995)

The theory of fractal geometry is applicable to many natural and mathematical phenomena, e. g., biological, geological, and atmospheric systems, dynamical systems, harmonic analysis, etc. It arises very naturally in work on signal and image processing. The three papers discussed below give extensions of both the theory and procedure of fractal geometry.

Self-Similar Fractal Sets : Theory and Procedure
IEEE Computer Graphics and Applications (May 1994)

Many natural objects and man-made processes exhibit intricate detail and scale invariance. Given some underlying geometry, these objects and processes can be studied under a branch of mathematics known as fractal geometry. This field was christened by B. Mandelbrot in the 1970's. He felt that the concepts studied under fractal geometry tied together his work and the related work of many researchers (including some 19th century scientists). Mandelbrot's book *The Fractal Geometry of Nature* was the first "textbook" in the field. It has been studied extensively, not only for its content, but also for its detailed bibliography and its pictures.

This article is written to complement Mandelbrot's book. It first gives a discussion of fractals and dimension theory. An efficient algorithm which produces approximations of self-similar fractal sets is then presented. The algorithm can be developed into a program which reproduces the self-similar fractals in Mandelbrot (approximately 45% of the figures in the book). The procedure is a

"pattern rewriting system" in which a given geometric pattern is drawn repeatedly after suitable mappings. Generating schemes are patterns with some built-in information on orientation in later levels of iteration. The figures presented in this article were generated by this program. We will then use some of the theory of fractal sets to calculate the dimensions of self–similar sets from their generators and describe images produced by the program.

The Interaction of Theory and Procedure in Fractal Geometry
Computer Graphics 1995 (March 1995)

This paper discusses the interplay in fractal geometry occurring between computer programs for developing (approximations of) fractal sets and the underlying dimension theory. The computer is ideally suited to implement the recursive algorithms needed to create these sets, thus giving us a laboratory for studying fractals and their corresponding dimensions. Moreover, this interaction between theory and procedure goes both ways. Dimension theory can be used to classify and understand fractal sets. This allows us, given a fixed generating pattern, to describe the resultant images produced by various programs. We will also tie these two perspectives in with the history of the subject. Three examples of fractal sets developed around the turn of the century are introduced and studied from both classical and modern viewpoints. Then, definitions and sample calculations of fractal and Hausdorff–Besicovitch dimension are given. Finally, dimension theory is used to classify images.

Five patterns with discussion for *The Pattern Book*
World Scientific Publishers (October 1995)

Five patterns were invited. Two were generated by the iteration of entire functions, which produced chaotic dynamics. Three additional patterns were produced by the "pattern rewriting system" described above.

4.) PERSONNEL SUPPORTED

All of the of the funding received under the grant (except for overhead) went to the support of Stephen Casey. This provided support during the summers of 1994 and 1995, and a computer. Work on projects associated to the grant was continued during AY 1994–95 (at American University), and will continue throughout AY 1995–96, during Casey's sabbatical year (at the Institute for Systems Research, University of Maryland). The grant also provided some hardware support for the research, in the form of a Dell Omniplex 560. Simulations of the theory were developed on this computer in MATLAB, Mathematica, and in Mathcad/Maple.

Additionally, during the period covered by the grant, Casey was the advisor to several graduate students. One of these students, Matthew Fivash, wrote the scholarly paper for his Masters degree on computer simulations of the multichannel deconvolution theory. During, this time, Fivash was working at the National Cancer Institute. He also received computational support, etc. from American University. Fivash's work is listed below. A second graduate student, Chris Organ, has just begun work on a related project involving irregular sampling.

5.) PUBLICATIONS

The following gives a list of publications over the duration of the grant.

1. (With N. Reingold) "Self–similar fractal sets: theory and procedure," *IEEE Computer Graphics and Applications*, 73–82 (May 1994) (Feature Article).

2. (With D. Walnut) "Systems of convolution equations, deconvolution, Shannon sampling, and the Gabor and wavelet transforms," *SIAM Review* **36** (4), 1–41 (December 1994).

3. (With B. Sadler) "Modifications of the euclidean algorithm for isolating periodicities from a sparse set of noisy measurements," accepted for publication in *IEEE Transactions on Signal Processing*, 32 pp.

4. (With B. Sadler) "A modified euclidean algorithm for isolating periodicities from noisy data," *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP '95)*, Vol. 3, pp. 1764–1767 (1995).

5. "The interaction of theory and procedure in fractal geometry," accepted for publication in *Computer Graphics 1995*, 16 pp. (1995).

6. Five patterns with discussion for *The Pattern Book*, edited by Dr. Clifford A. Pickover, IBM T. J. Watson Research Center. Two of these are joint work with N. Reingold. 9 pages, 6 figures – World Scientific Publishers (1995).

7. "Bezout equations, multichannel deconvolution and sampling," submitted to *Inverse Problems*, 24 pp.

8. (With B. Sadler) "On pulse interval analysis with outliers and missing observations," submitted to *IEEE Transactions on Signal Processing*, 26 pp.

9. (With B. Sadler) "PRI analysis from sparse data via a modified Euclidean algorithm," invited paper for 29th *Annual Asilomar Conference on Signals, Systems, and Computers*, 6 pp.

10. (With B. Sadler) "Frequency estimation via sparse zero crossings," accepted for *IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP '96)*, 4 pp.

The following items are papers in progress.

1. "Optimal multichannel deconvolution in a noisy environment: deconvolving Wiener filters," to be submitted to *Fourier Analysis and Applications*, 38 pp.

2. "Multichannel deconvolution, splines and sampling," to be submitted to *Applied and Computational Harmonic Analysis*, 44 pp.

3. "Radios, radar, and the zeta function of Riemann: with apologies to Professor Hardy," to be submitted to *Mathematics Magazine*, 46 pp.

4. (With M. Fivash) "Multichannel deconvolution: theory and procedure."

5. "New methods for the inversion of Radon transforms."

The following items are projects completed under my direction.

1. "Multichannel deconvolution: a computational study," by Matthew Fivash, scholarly paper for the Masters Degree, American University, 41 pp. (January 1995).

2. "Multichannel deconvolution: a computational study," by Matthew Fivash, submitted to the *SIAM Student Paper Contest*, 18 pp. (June 1995).

6.) INTERACTIONS/TRANSITIONS

Presentations

1. 1996 International Conference on Acoustics, Speech, and Signal Processing, Atlanta – "Frequency estimation via sparse zero–crossings" – invited for May 1996

2. Twenty–Ninth Annual Asilomar Conference on Signals, Systems, and Computers, Pacific Grove, California – "PRI analysis from sparse data via a modified Euclidean algorithm" – October 1995

3. Conference on Complex Interpolation Theory and Applications, George Mason University – "Multichannel deconvolution and Wiener filtering" – October 1995

4. 1995 International Conference on Acoustics, Speech, and Signal Processing, Detroit – "A modified Euclidean algorithm for isolating periodicities from a sparse set of noisy measurements" – May 1995

5. Meeting of the American Mathematical Society, University of Central Florida – special session on "Wavelets in Sampling Theory and Signal Processing" – "An algorithm for isolating periodicities from noisy data" – March 1995

6. Invited paper submitted for 1995 Winter School of Computer Graphics, University of West Bohemia, Plsen. Czech Republic – "The interaction of theory and procedure in fractal geometry" – February 1995

7. Army Research Laboratory, Optical and Digital Signal Processing Branches – "Exact deconvolution from a system of convolution equations with applications to signal and image processing" – January 1995

8. Maryland Undergraduate Mathematics Enhancement Program, Visual Thinking in Fractal Geometry, – "Dimension theory and procedure in fractal geometry" – October 1994

9. Maryland Undergraduate Mathematics Enhancement Program, Visual Thinking in Fractal Geometry, – "An efficient algorithm for producing self–similar fractal sets" – July 1994

10. Joint Meeting of the American Mathematical Society and the Mathematical Association of America, Cincinnati, OH – special session on "Advances in Function Theoretic Methods" – "Bezout equations and sampling" – January 1994

Consultations

The work on parameter estimation was co–authored with Dr. Brian Sadler of the Army Research Laboratory, Adelphi. MD. I met Sadler when I worked at the Labs (then Harry Diamond Labs) from 1984–88. I had developed the mathematics basis for the modified Euclidean algorithms, and was looking for a broad class of problems to which the theory was applicable. Sadler suggested

applications to radar and communications systems. The resulting work was very well received at *ICASSP '95*, where it was suggested that modifications would work on a broad class of PRI problems in radar. This resulted in the work on deinterleaving and the work on spectral analysis of time–series data from a sparse set of zero–crossings.

The work on parameter estimation has also drawn the interest of Dr. James Tsui at the Wright Laboratory at Wright–Patterson AFB. I have mailed copies of the first *IEEE Signal Processing* paper and the *ICASSP '95* paper to him.

Transitions

None.

7.) NEW DISCOVERIES

The work on multichannel deconvolution is certainly a new perspective on this class of problems. At this stage, the work is theoretical; no actual multichannel systems have been developed. The success of the simulation work, however, gives proof that it would not be difficult to develop such a system.

The work on parameter estimation gives new computationally straightforward algorithms for estimating the fundamental period from sparse, noisy data. Further work has developed algorithms for deinterleaving data that is a combination of several periodic processes. Robust versions are developed that are stable despite significant jitter noise and the presence of arbitrary outliers. These problem arises in radar pulse repetition interval (PRI) analysis, in bit synchronization in communications, and many other scenarios. These algorithms have been extensively simulated, and can be easily implemented into actual systems to perform this analysis. This work is directly applicable to AF technologies in that it provides new procedures for period estimation and/or the deinterleaving of radar and sonar data.